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Existence results for regularized equations of second-grade fluids with wall slip

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Abstract. In this paper, we study the equations governing the steady motion of a class of second-grade fluids in a bounded domain of $\mathbb{R}^n$, $n = 2, 3$, with the nonlinear slip boundary condition. We prove the existence of a weak solution without assuming smallness of the data. Moreover, we give estimates for weak solutions and show that the solution set is sequentially weakly closed.

Keywords: nonlinear PDE, existence theorem, operator theory, weak solutions, second-grade fluids, slip boundary conditions.

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1 Introduction

Experimental and theoretical investigations show that non-Newtonian fluids often exhibit wall slip, and that this is governed by nonlinear relation between the slip velocity and the shear stress (see, e.g., [10,14] and the references therein). In this paper, we investigate the boundary value problem for the regularized stationary equations of motion of second-grade fluids with the nonlinear slip boundary condition.

Steady flows of an incompressible fluid are described by the system of differential equations in the Cauchy form [13]:

$$\rho \sum_{j=1}^{n} v_j \frac{\partial v}{\partial x_j} = -\nabla p + \operatorname{Div} \mathbf{S} + \rho f \quad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{div} v = 0 \quad \text{in } \Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is the flow domain, $v = v(x)$ is the velocity vector of the particle at a point $x \in \Omega$, $p = p(x)$ is the pressure, ∇ denotes Eulerian spatial gradient, $f = f(x)$ is the external body force, ρ is the positive constant density (without loss of generality it can be assumed that $\rho = 1$), $\mathbf{S} = \mathbf{S}(x)$ is the extra stress that is specified constitutively. The divergence $\operatorname{Div} \mathbf{S}$ is the vector with coordinates

$$(\operatorname{Div} \mathbf{S})_j = \sum_{i=1}^{n} \frac{\partial S_{ij}}{\partial x_i}.$$

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For fluids of second-grade (see [23]) the extra stress $\mathbf{S}$ is given by

$$\mathbf{S} = \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2. \tag{1.3}$$

Here, μ is the viscosity, α_1 and α_2 are the normal stress moduli, $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first two Rivlin–Ericksen tensors:

$$\begin{aligned} \mathbf{A}_1 &= \mathbf{A}_1(\boldsymbol{v}) := \nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T, \\ \mathbf{A}_2 &= \mathbf{A}_2(\boldsymbol{v}) := D_t \mathbf{A}_1 + \mathbf{A}_1(\nabla \boldsymbol{v}) + (\nabla \boldsymbol{v})^T \mathbf{A}_1 \\ &= \sum_{j=1}^{n} v_j \frac{\partial \mathbf{A}_1(\boldsymbol{v})}{\partial x_j} + \mathbf{A}_1(\nabla \boldsymbol{v}) + (\nabla \boldsymbol{v})^T \mathbf{A}_1. \end{aligned}$$

It was shown in [11,12] that

$$\mu \geq 0, \qquad \alpha_1 \geq 0, \qquad \alpha_1 + \alpha_2 = 0.$$

Let us denote

$$\alpha := \alpha_1 = -\alpha_2, \tag{1.4}$$

$$\mathbf{W}(\boldsymbol{v}) := \frac{1}{2}\big(\nabla \boldsymbol{v} - (\nabla \boldsymbol{v})^T\big). \tag{1.5}$$

The tensor $\mathbf{W}$ is called the vorticity tensor.

Combining (1.3), (1.4), and (1.5), we get

$$\mathbf{S} = \mu \mathbf{A}_1 + \alpha \sum_{j=1}^{n} v_j \frac{\partial \mathbf{A}_1(\boldsymbol{v})}{\partial x_j} + \alpha\big(\mathbf{A}_1(\boldsymbol{v})\mathbf{W}(\boldsymbol{v}) - \mathbf{W}(\boldsymbol{v})\mathbf{A}_1(\boldsymbol{v})\big).$$

Following [28], we consider the regularized constitutive law

$$\mathbf{S} = \mu \mathbf{A}_1 + \alpha \sum_{j=1}^{n} v_j \frac{\partial \mathbf{A}_1(\boldsymbol{v})}{\partial x_j} + \alpha\big(\mathbf{A}_1(\boldsymbol{v})\mathbf{W}_\rho(\boldsymbol{v}) - \mathbf{W}_\rho(\boldsymbol{v})\mathbf{A}_1(\boldsymbol{v})\big). \tag{1.6}$$

Here, $\mathbf{W}_\rho$ is the regularized vorticity tensor:

$$\mathbf{W}_\rho(\boldsymbol{v})(\boldsymbol{x}) = \int_{\mathbb{R}^n} \rho(\boldsymbol{x} - \boldsymbol{y}) \mathbf{W}(\boldsymbol{v})(\boldsymbol{y}) d\boldsymbol{y}, \tag{1.7}$$

$\rho\colon \mathbb{R}^n \to \mathbb{R}$ is a smooth function with compact support such that

$$\int_{\mathbb{R}^n} \rho(\boldsymbol{x}) d\boldsymbol{x} = 1$$

and $\rho(\boldsymbol{x}) = \rho(\boldsymbol{y})$ whenever $|\boldsymbol{x}| = |\boldsymbol{y}|$. In formula (1.7), we set $\mathbf{W}(\boldsymbol{v})(\boldsymbol{y}) = \mathbf{0}$ if $\boldsymbol{y} \in \mathbb{R}^n \setminus \Omega$.

Note that the constitutive law (1.6) is frame-indifferent. This means (see e.g. [27]) that the form of (1.6) does not change after a change of spatial variables. This can be proved by methods of [28].

We will use general Navier-type slip boundary conditions (see [21] for the original reference):

$$\boldsymbol{v} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \tag{1.8}$$

$$(\mathbf{S}\mathbf{n})_\tau = -\lambda(\boldsymbol{x}, |\boldsymbol{v}_\tau|)\boldsymbol{v}_\tau \quad \text{on } \Gamma, \tag{1.9}$$

where Γ is the boundary of the flow region, $\mathbf{n}$ is the outer unit normal on Γ, $\boldsymbol{v}\cdot\mathbf{n}$ is the scalar product of the vectors $\boldsymbol{v}$ and $\mathbf{n}$ in space $\mathbb{R}^3$, $\boldsymbol{u}_\tau$ denotes the tangential component of any vector field $\boldsymbol{u}$ defined on Γ, i.e.,

$$\boldsymbol{u}_\tau = \boldsymbol{u} - (\boldsymbol{u}\cdot\mathbf{n})\,\mathbf{n},$$

and the slip coefficient λ is a given function $\lambda\colon \Gamma\times[0,\infty)\to[0,\infty)$.

Boundary value problems for the equations of motion of second-grade fluids have been studied by several authors. Existence results for the corresponding no-slip problems have been established under various restrictions on the data in [5,7–9,20]. In [2], the optimal flow control problem for second-grade fluids was studied. The slip problem for time-dependent flows of second-grade fluids was considered by Le Roux in [17]. He showed the existence of a unique classical solution under suitable regularity and growth restrictions on the data and the body and surface forces. For small initial data, the global existence of H^3 solutions is shown in Busuioc and Ratiu [6]. Moreover, Tani and Le Roux [25] proved the unique solvability in Hölder spaces of the stationary slip problem with a sufficiently small body force.

In this paper, we consider the slip problem without restrictions on the data. Instead of assuming smallness of the data, we use the regularization of constitutive law. We show the existence of a weak solution and establish some estimates for the flow velocity. We also show that the solution set is sequentially weakly closed.

To prove the existence of weak solutions, we give the operator treatment for the slip problem and investigate the corresponding operator equation. First we construct a suitable system of functional spaces with a special basis. Then we construct approximate solutions by the Galerkin method. Using the special properties of the basis, we show that the limit of a sequence of approximate solutions is a solution of the original problem.

Note that the proposed approach is adapted for the study of non-homogeneous boundary value problems. The classical methods are not always effective for such problems. For example, Oskolkov [22] (see also [29]) proved the existence of a weak solution for a simplified version of (1.1)–(1.3) with homogeneous boundary data. He used the method of introduction of auxiliary viscosity. However, the variant with Navier slip boundary condition produces significant technical obstacles. The integration by parts of auxiliary viscosity terms produces additional terms in the motion equations. These terms do not vanish when the regularization parameter tends to zero, and one has to find new methods for the study of non-homogeneous boundary value problems. One of such methods is presented in this article.

2 Statement of the problem and the main result

Let Ω be a bounded locally Lipschitz domain in space $\mathbb{R}^n$, $n=2,3$, with boundary Γ. Consider the following boundary value problem:

$$\sum_{j=1}^{n} v_j\frac{\partial \boldsymbol{v}}{\partial x_j} - \mu\Delta\boldsymbol{v} - \alpha\,\mathrm{Div}\left(\sum_{j=1}^{n} v_j\frac{\partial \mathbf{A}_1(\boldsymbol{v})}{\partial x_j}\right) - \alpha\,\mathrm{Div}\left(\mathbf{A}_1(\boldsymbol{v})\mathbf{W}_\rho(\boldsymbol{v}) - \mathbf{W}_\rho(\boldsymbol{v})\mathbf{A}_1(\boldsymbol{v})\right) + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{2.1}$$

$$\mathrm{div}\,\boldsymbol{v} = 0 \quad \text{in } \Omega, \tag{2.2}$$

$$\boldsymbol{v}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma, \tag{2.3}$$

$$\left(\left[\mu \mathbf{A}_1(v) + \alpha\left(\sum_{j=1}^{n} v_j \frac{\partial \mathbf{A}_1(v)}{\partial x_j} + \mathbf{A}_1(v)\mathbf{W}_\rho(v) - \mathbf{W}_\rho(v)\mathbf{A}_1(v)\right)\right]\mathbf{n}\right)_\tau = -\lambda(\boldsymbol{x}, |v_\tau|)v_\tau \quad \text{on } \Gamma. \quad (2.4)$$

Remark 2.1. Equation (2.1) is obtained from relations (1.1) and (1.6). Boundary condition (2.4) is obtained from (1.9) and (1.6).

Let us describe the concept of a weak solution (see also [3,4,19]).

We use the standard notations $\mathbf{L}_p(\Omega)$, $\mathbf{H}^m(\Omega)$, $\mathbf{H}^{m-1/2}(\Gamma)$ for Lebesgue and Sobolev spaces of vector-functions. The scalar product in $\mathbf{L}_2(\Omega)$ is denoted by $(\cdot,\cdot)$. The restriction of $v \in \mathbf{H}^1(\Omega)$ to Γ is given by the formula $v|_\Gamma = \gamma_0 v$, where $\gamma_0 \colon \mathbf{H}^1(\Omega) \to \mathbf{H}^{1/2}(\Gamma)$ is the trace operator (see, e.g., [18]).

We set

$$\begin{aligned}
&\mathcal{X}(\Omega) = \{v \in \mathbf{C}^\infty(\bar{\Omega}) : \operatorname{div} v = 0,\ v|_\Gamma \cdot \mathbf{n} = 0\},\\
&\boldsymbol{X}(\Omega) = \text{the closure of } \mathcal{X}(\Omega) \text{ in } \mathbf{H}^1(\Omega),\\
&[\boldsymbol{X}(\Omega)]^* \text{ is the dual space of } \boldsymbol{X}(\Omega).
\end{aligned}$$

Suppose the function $\lambda \colon \Gamma \times [0,\infty) \to [0,\infty)$ satisfies the condition

$$0 < \lambda_0 \le \lambda(\boldsymbol{x}, \xi) \le \lambda_1, \qquad (\boldsymbol{x}, \xi) \in \Gamma \times [0,\infty),$$

where λ_0 and λ_1 are constants.

Define the scalar product in $\boldsymbol{X}(\Omega)$ as

$$(v, w)_{\mathbf{X}(\Omega)} = \frac{\mu}{2}(\mathbf{A}_1(v), \mathbf{A}_1(w)) + \lambda_0 \int_\Gamma (v_\tau \cdot w_\tau)\, \mathrm{d}\Gamma.$$

The scalar product in well defined. This follows from Korn's inequality (see, e.g., [19, Chapter II, Theorems 2.2 and 2.3]):

$$\|\mathbf{A}_1(v)\|^2_{\mathbf{L}_2(\Omega)} + \|\gamma_0 v\|^2_{\mathbf{L}_2(\Gamma)} \ge C\|v\|^2_{\mathbf{H}^1(\Omega)}, \qquad v \in \mathbf{H}^1(\Omega),$$

where C is a constant.

Suppose that $\boldsymbol{f} \in \mathbf{L}_2(\Omega)$.

Definition 2.2. We shall say that a function $v \in \boldsymbol{X}(\Omega)$ is a *weak solution* of problem (2.1)–(2.4) if the equality

$$-\sum_{j=1}^{n}\left(v_j v, \frac{\partial \boldsymbol{\varphi}}{\partial x_j}\right) + \frac{\mu}{2}(\mathbf{A}_1(v), \mathbf{A}_1(\boldsymbol{\varphi})) - \frac{\alpha}{2}\sum_{j=1}^{n}\left(\mathbf{A}_1(v), v_j \frac{\partial \mathbf{A}_1(\boldsymbol{\varphi})}{\partial x_j}\right) + \frac{\alpha}{2}(\mathbf{A}_1(v)\mathbf{W}_\rho(v) - \mathbf{W}_\rho(v)\mathbf{A}_1(v), \mathbf{A}_1(\boldsymbol{\varphi})) + \int_\Gamma \lambda(\boldsymbol{x}, |v_\tau|)v_\tau \cdot \boldsymbol{\varphi}_\tau \, d\Gamma = (\boldsymbol{f}, \boldsymbol{\varphi}) \quad (2.5)$$

holds for any $\boldsymbol{\varphi} \in \mathcal{X}(\Omega)$.

Remark 2.3. Equality (2.5) appears from the following reasoning. Let (v, p) be a classical solution of problem (2.1)–(2.4). We can rewrite (2.1), (2.4) as follows

$$\sum_{j=1}^{n} v_j \frac{\partial v}{\partial x_j} - \operatorname{Div} \mathbf{S} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \quad (2.6)$$

$$\left(\mathbf{Sn}\right)_\tau = -\lambda(\boldsymbol{x}, |\boldsymbol{v}_\tau|)\boldsymbol{v}_\tau \quad \text{on } \Gamma, \tag{2.7}$$

where

$$\mathbf{S} = \mu \mathbf{A}_1(\boldsymbol{v}) + \alpha \sum_{j=1}^{n} v_j \frac{\partial \mathbf{A}_1(\boldsymbol{v})}{\partial x_j} + \alpha\left(\mathbf{A}_1(\boldsymbol{v})\mathbf{W}_\rho(\boldsymbol{v}) - \mathbf{W}_\rho(\boldsymbol{v})\mathbf{A}_1(\boldsymbol{v})\right). \tag{2.8}$$

Taking the L_2-scalar product of equality (2.6) with $\boldsymbol{\varphi} \in \mathcal{X}(\Omega)$, we obtain

$$\sum_{j=1}^{n} \left(v_j \frac{\partial \boldsymbol{v}}{\partial x_j}, \boldsymbol{\varphi} \right) - (\operatorname{Div} \mathbf{S}, \boldsymbol{\varphi}) + (\nabla p, \boldsymbol{\varphi}) = (\boldsymbol{f}, \boldsymbol{\varphi}). \tag{2.9}$$

Integrating by parts the terms of equality (2.9), we have

$$-\sum_{j=1}^{n} \left(v_j \boldsymbol{v}, \frac{\partial \boldsymbol{\varphi}}{\partial x_j} \right) + (\mathbf{S}, \nabla \boldsymbol{\varphi}) - \int\limits_{\Gamma} (\mathbf{Sn}) \cdot \boldsymbol{\varphi} \, d\Gamma = (\boldsymbol{f}, \boldsymbol{\varphi}). \tag{2.10}$$

Since the matrix $\mathbf{S}$ is symmetric, it follows that

$$\begin{aligned} (\mathbf{S}, \nabla \boldsymbol{\varphi}) &= \frac{1}{2} (\mathbf{S}, \nabla \boldsymbol{\varphi}) + \frac{1}{2} \left(\mathbf{S}^T, (\nabla \boldsymbol{\varphi})^T \right) \\ &= \frac{1}{2} (\mathbf{S}, \nabla \boldsymbol{\varphi}) + \frac{1}{2} \left(\mathbf{S}, (\nabla \boldsymbol{\varphi})^T \right) = \frac{1}{2} (\mathbf{S}, \mathbf{A}_1(\boldsymbol{\varphi})). \end{aligned} \tag{2.11}$$

Furthermore, taking into account

$$\boldsymbol{\varphi}|_\Gamma \cdot \mathbf{n} = 0,$$

we obtain

$$\int\limits_{\Gamma} (\mathbf{Sn}) \cdot \boldsymbol{\varphi} \, d\Gamma = \int\limits_{\Gamma} (\mathbf{Sn})_\tau \cdot \boldsymbol{\varphi}_\tau \, d\Gamma. \tag{2.12}$$

Using (2.11) and (2.12), we can rewrite (2.10) in the following form:

$$-\sum_{j=1}^{n} \left(v_j \boldsymbol{v}, \frac{\partial \boldsymbol{\varphi}}{\partial x_j} \right) + \frac{1}{2} (\mathbf{S}, \mathbf{A}_1(\boldsymbol{\varphi})) - \int\limits_{\Gamma} (\mathbf{Sn})_\tau \cdot \boldsymbol{\varphi}_\tau \, d\Gamma = (\boldsymbol{f}, \boldsymbol{\varphi}). \tag{2.13}$$

Combining (2.13), (2.7), and (2.8), we get equality (2.5).

Remark 2.4. Let us show that if the weak solution $\boldsymbol{v}$ of problem (2.1)–(2.4) is sufficiently smooth, then there exists a function p such that $(\boldsymbol{v}, p)$ is a classical solution. Multiplying (2.5) by -1 and integrating by parts, we can rewrite (2.5) as follows:

$$\left(-\sum_{j=1}^{n} v_j \frac{\partial \boldsymbol{v}}{\partial x_j} + \operatorname{Div} \boldsymbol{S} + \boldsymbol{f}, \boldsymbol{\varphi} \right) = \int\limits_{\Gamma} \left((\mathbf{Sn})_\tau + \lambda(\boldsymbol{x}, |\boldsymbol{v}_\tau|)\boldsymbol{v}_\tau \right) \cdot \boldsymbol{\varphi}_\tau \, d\Gamma \tag{2.14}$$

with $\boldsymbol{S}$ defined in (1.6). This yields that

$$\left(-\sum_{j=1}^{n} v_j \frac{\partial \boldsymbol{v}}{\partial x_j} + \operatorname{Div} \boldsymbol{S} + \boldsymbol{f}, \boldsymbol{\psi} \right) = 0$$

for any $\boldsymbol{\psi} \in \mathbf{H}^1(\Omega)$ such that $\operatorname{div} \boldsymbol{\psi} = 0$ and $\boldsymbol{\psi}|_\Gamma = \mathbf{0}$. Hence (see, e.g., [16]), there exists a function p such that

$$-\sum_{j=1}^{n} v_j \frac{\partial \boldsymbol{v}}{\partial x_j} + \operatorname{Div} \boldsymbol{S} + \boldsymbol{f} = \nabla p. \tag{2.15}$$

This implies that equation (2.1) holds. Moreover, by definition of the space $\boldsymbol{X}(\Omega)$, equalities (2.2) and (2.3) are valid. It remains to show that boundary condition (2.4) holds. Substituting (2.15) in (2.14), we get

$$(\nabla p, \boldsymbol{\varphi}) = \int\limits_{\Gamma} \big((\mathbf{S}\mathbf{n})_\tau + \lambda(\boldsymbol{x}, |\boldsymbol{v}_\tau|)\boldsymbol{v}_\tau\big) \cdot \boldsymbol{\varphi}_\tau \, d\Gamma \tag{2.16}$$

for any $\boldsymbol{\varphi} \in \boldsymbol{X}(\Omega)$. Integrating by parts, we obtain that the left-hand side of (2.16) is equal to zero. Therefore, we have

$$\int\limits_{\Gamma} \big((\mathbf{S}\mathbf{n})_\tau + \lambda(\boldsymbol{x}, |\boldsymbol{v}_\tau|)\boldsymbol{v}_\tau\big) \cdot \boldsymbol{\varphi}_\tau \, d\Gamma = 0. \tag{2.17}$$

Since the set $\{\boldsymbol{\varphi}|_\Gamma : \boldsymbol{\varphi} \in \boldsymbol{X}(\Omega)\}$ is dense in the space

$$\{\boldsymbol{w} \in \mathbf{L}_2(\Gamma) : \ \boldsymbol{w} \cdot \mathbf{n} = 0\},$$

it follows that equality (2.17) still holds by continuity for any vector function $\boldsymbol{\varphi} \in \mathbf{L}_2(\Gamma)$ such that $\boldsymbol{\varphi} \cdot \mathbf{n} = 0$. This implies that

$$(\mathbf{S}\mathbf{n})_\tau = -\lambda(\boldsymbol{x}, |\boldsymbol{v}_\tau|)\boldsymbol{v}_\tau \text{ on } \Gamma,$$

i.e., boundary condition (2.4) holds.

The main result of this paper is the following.

Theorem 2.5.

1) *Boundary value problem* (2.1)–(2.4) *has a weak solution, which satisfies the estimate*

$$\frac{\mu}{2}\|\mathbf{A}_1(\boldsymbol{v})\|^2_{\mathbf{L}_2(\Omega)} + \lambda_0 \|\boldsymbol{v}_\tau\|^2_{\mathbf{L}_2(\Gamma)} \le \|\boldsymbol{f}\|^2_{[\boldsymbol{X}(\Omega)]^*}.$$

2) *The weak solution set is sequentially weakly closed.*

The proof of Theorem 2.5 is given in Section 4.

3 Auxiliary results

To prove Theorem 2.5, we study a class of operator equations.

Let $\boldsymbol{F}_0, \boldsymbol{F}_1, \boldsymbol{Z}_1, \dots, \boldsymbol{Z}_m$ be separable real Hilbert spaces, and $\mathcal{X}$ be a linear subspace of $\boldsymbol{F}_0$.

Suppose the embedding $\boldsymbol{F}_0 \subset \boldsymbol{F}_1$ is continuous. Let symbols $\boldsymbol{X}$ and $\boldsymbol{Y}$ denote the closures of $\mathcal{X}$ in $\boldsymbol{F}_1$ and $\boldsymbol{F}_0$, respectively.

Let $\boldsymbol{Z}_i^*$ be the set of all continuous linear functionals on $\boldsymbol{Z}_i$. The value of a functional from $\boldsymbol{Z}_i^*$ on an element from $\boldsymbol{Z}_i$ is denoted by $\langle \cdot, \cdot \rangle$.

Let $\boldsymbol{T}_i \colon \boldsymbol{X} \to \boldsymbol{Z}_i^*$, $\boldsymbol{Q}_i \colon \boldsymbol{X} \times \boldsymbol{Y} \to \boldsymbol{Z}_i$ be nonlinear operators. Fix a functional $\boldsymbol{f} \in \boldsymbol{X}^*$.

We assume the following:

(i) the map $\boldsymbol{Q}_i[\boldsymbol{v}, \cdot] \colon \boldsymbol{Y} \to \boldsymbol{Z}_i$ is linear for any $\boldsymbol{v} \in \mathcal{X}$, $i = 1, \dots, m$,

(ii) there exists a function $a \colon [0, \infty) \to \mathbb{R}$ such that

$$\sum_{i=1}^{m} \langle \boldsymbol{T}_i(\boldsymbol{v}), \boldsymbol{Q}_i[\boldsymbol{v}, \boldsymbol{v}] \rangle \ge a(\|\boldsymbol{v}\|_{\boldsymbol{X}}), \qquad \boldsymbol{v} \in \mathcal{X},$$

(iii) there exists a constant $r_0 > 0$ such that

$$\frac{a(r_0)}{r_0} > \|f\|_{X^*},$$

(iv) for any sequence $\{v_k\} \subset X$ such that $v_k \to v_0$ weakly in X, it follows that

$$\langle T_i(v_k), Q_i[v_k, \varphi] \rangle \to \langle T_i(v_0), Q_i[v_0, \varphi] \rangle, \qquad \varphi \in \mathcal{X},\ i = 1, \dots, m$$

as $k \to \infty$.

Consider the following problem: *Find an element $v \in X$ such that*

$$\sum_{i=1}^{m} \langle T_i(v), Q_i[v, \varphi] \rangle = \langle f, \varphi \rangle \tag{A}$$

for any $\varphi \in \mathcal{X}$.

Theorem 3.1. *Under conditions* (i)–(iv), *there exist a solution to problem* (A) *in the ball $\{v \in X : \|v\|_X \le r_0\}$.*

Proof. Consider a sequence

$$(x_1, y_1), (x_2, y_2), \dots, (x_k, y_k), \dots$$

such that $(x_i, y_i) \in X \times Y$ for any $i \in \mathbb{N}$, the set $\{x_1, x_2, \dots, x_k, \dots\}$ is everywhere dense in the space X, and the set $\{y_1, y_2, \dots, y_k, \dots\}$ is everywhere dense in the space Y.

Since the set $\mathcal{X} \times \mathcal{X}$ is everywhere dense in $X \times Y$, we see that there exist a pair $(\varphi_{11}, \psi_{11}) \in \mathcal{X} \times \mathcal{X}$ such that

$$\|x_1 - \varphi_{11}\|_X \le \frac{1}{2}, \qquad \|y_1 - \psi_{11}\|_Y \le \frac{1}{2}.$$

Moreover, there are pairs $(\varphi_{21}, \psi_{21}), (\varphi_{22}, \psi_{22}) \in \mathcal{X} \times \mathcal{X}$ such that

$$\|x_1 - \varphi_{21}\|_X \le \frac{1}{4}, \qquad \|y_1 - \psi_{21}\|_Y \le \frac{1}{4},$$

$$\|x_2 - \varphi_{22}\|_X \le \frac{1}{4}, \qquad \|y_2 - \psi_{22}\|_Y \le \frac{1}{4}.$$

Similarly, for each $k \in \mathbb{N}$, there are k pairs

$$(\varphi_{k1}, \psi_{k1}), \dots, (\varphi_{kk}, \psi_{kk}) \in \mathcal{X} \times \mathcal{X}$$

such that

$$\|x_i - \varphi_{ki}\|_X \le \frac{1}{2^k}, \qquad \|y_i - \psi_{ki}\|_Y \le \frac{1}{2^k}, \qquad i = 1, \dots, k.$$

Consider the following sequence

$$\varphi_{11},\ \psi_{11},\ \varphi_{21},\ \psi_{21},\ \varphi_{22},\ \psi_{22},\ \varphi_{31},\ \psi_{31}, \dots \tag{3.1}$$

It is clear that system (3.1) is complete in the space X, i.e., the smallest closed subspace containing (3.1) is the whole space X. Moreover, system (3.1) is complete in Y.

We make the following transformations of sequence (3.1):

1) we eliminate from sequence (3.1) all elements which can be written as linear combinations of elements with smaller indices;

2) we apply the orthogonalization process (in $\boldsymbol{X}$) to the obtained system.

Denote the resulting system as $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_j, \dots$ We get an orthonormal basis in $\boldsymbol{X}$ such that $\boldsymbol{u}_j \in \mathcal{X}$ for any $j \in \mathbb{N}$ and the system $\{\boldsymbol{u}_j\}_{j=1}^{\infty}$ is complete in $\boldsymbol{Y}$.

We fix a number $k \in \mathbb{N}$. Consider the auxiliary problem:

Find a vector $\boldsymbol{\xi} = (\xi_1, \dots, \xi_k) \in \mathbb{R}^k$ such that

$$\sum_{i=1}^{m} \langle \boldsymbol{T}_i(\boldsymbol{v}_k), \boldsymbol{Q}_i[\boldsymbol{v}_k, \boldsymbol{u}_j] \rangle = \langle \boldsymbol{f}, \boldsymbol{u}_j \rangle, \qquad j = 1, \dots, k, \tag{3.2}$$

$$\boldsymbol{v}_k = \sum_{i=1}^{k} \xi_i \boldsymbol{u}_i. \tag{3.3}$$

Let us show that problem (3.2), (3.3) is solvable. Let us define the operator $\boldsymbol{G}_k \colon \mathbb{R}^k \to \mathbb{R}^k$ by

$$\boldsymbol{G}_k(\boldsymbol{\xi}) = \left(\sum_{i=1}^{m} \langle \boldsymbol{T}_i(\boldsymbol{v}_k), \boldsymbol{Q}_i[\boldsymbol{v}_k, \boldsymbol{u}_1] \rangle, \dots, \sum_{i=1}^{m} \langle \boldsymbol{T}_i(\boldsymbol{v}_k), \boldsymbol{Q}_i[\boldsymbol{v}_k, \boldsymbol{u}_k] \rangle \right)$$

and the vector $\boldsymbol{f}_k \in \mathbb{R}^k$, $\boldsymbol{f}_k = \big(\langle \boldsymbol{f}, \boldsymbol{u}_1 \rangle, \dots, \langle \boldsymbol{f}, \boldsymbol{u}_k \rangle\big)$.

It is clear that problem (3.2), (3.3) is equivalent to the operator equation

$$\boldsymbol{G}_k(\boldsymbol{\xi}) = \boldsymbol{f}_k. \tag{3.4}$$

Observe that

$$(\boldsymbol{G}_k(\boldsymbol{\xi}) - \boldsymbol{f}_k, \boldsymbol{\xi})_{\mathbb{R}^k} > 0$$

for all $\boldsymbol{\xi} \in \mathbb{R}^k$ such that $\|\boldsymbol{\xi}\|_{\mathbb{R}^k} = r_0$. In fact, using (i)–(iii), we get

$$\begin{aligned}
(\boldsymbol{G}_k(\boldsymbol{\xi}) - \boldsymbol{f}_k, \boldsymbol{\xi})_{\mathbb{R}^n} &= \sum_{j=1}^{k} \sum_{i=1}^{m} \langle \boldsymbol{T}_i(\boldsymbol{v}_k), \boldsymbol{Q}_i[\boldsymbol{v}_k, \boldsymbol{u}_j] \rangle \xi_j - (\boldsymbol{f}_k, \boldsymbol{\xi})_{\mathbb{R}^k} \\
&= \sum_{i=1}^{m} \langle \boldsymbol{T}_i(\boldsymbol{v}_k), \boldsymbol{Q}_i[\boldsymbol{v}_k, \boldsymbol{v}_k] \rangle - (\boldsymbol{f}_k, \boldsymbol{\xi})_{\mathbb{R}^k} \geq a(\|\boldsymbol{v}_k\|_{\boldsymbol{X}}) - \|\boldsymbol{f}\|_{\boldsymbol{X}^*} \|\boldsymbol{\xi}\|_{\mathbb{R}^k} \\
&= \big(a(\|\boldsymbol{\xi}\|_{\mathbb{R}^k}) / \|\boldsymbol{\xi}\|_{\mathbb{R}^k} - \|\boldsymbol{f}\|_{\boldsymbol{X}^*}\big) \|\boldsymbol{\xi}\|_{\mathbb{R}^k} = \big(a(r_0)/r_0 - \|\boldsymbol{f}\|_{\boldsymbol{X}^*}\big) r_0 > 0.
\end{aligned}$$

Then, by [26, Chapter II, Lemma 1.4], equation (3.4) (and, therefore, problem (3.2), (3.3)) has a solution in the ball $\{\boldsymbol{\xi} \in \mathbb{R}^k : \ \|\boldsymbol{\xi}\|_{\mathbb{R}^k} \leq r_0\}$.

Let $\boldsymbol{\xi}_k$ be a solution of (3.4). Consider the vector $\boldsymbol{v}_k$ defined by formula (3.3). It is obvious that

$$\|\boldsymbol{v}_k\|_{\boldsymbol{X}} = \|\boldsymbol{\xi}_k\|_{\mathbb{R}^k} \leq r_0. \tag{3.5}$$

Since r_0 does not depend on k, there exist a element $\boldsymbol{v}_* \in \boldsymbol{X}$ such that $\boldsymbol{v}_{k_j} \to \boldsymbol{v}_*$ weakly in $\boldsymbol{X}$ for some subsequence $\{k_j\}$. We can assume without loss of generality that $\boldsymbol{v}_k \to \boldsymbol{v}_*$ weakly in $\boldsymbol{X}$.

Using (3.2) and (iv), we conclude that

$$\sum_{i=1}^{m} \langle \boldsymbol{T}_i(\boldsymbol{v}_*), \boldsymbol{Q}_i[\boldsymbol{v}_*, \boldsymbol{u}_j] \rangle = \langle \boldsymbol{f}, \boldsymbol{u}_j \rangle$$

for any $j \in \mathbb{N}$. The system $\{\boldsymbol{u}_j\}_{j=1}^{\infty}$ is complete in the space $\boldsymbol{Y}$. This yields that

$$\sum_{i=1}^{m} \langle \boldsymbol{T}_i(\boldsymbol{v}_*), \boldsymbol{Q}_i[\boldsymbol{v}_*, \boldsymbol{\varphi}] \rangle = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle$$

for any $\boldsymbol{\varphi} \in \mathcal{X}$. Thus, the element $\boldsymbol{v}_*$ is a solution to problem (A).

Let us recall that $\boldsymbol{v}_k \to \boldsymbol{v}_*$ weakly in $\boldsymbol{X}$. Taking into account (3.5), we obtain $\|\boldsymbol{v}_*\|_{\boldsymbol{X}} \leq r_0$. This completes the proof. □

4 Proof of Theorem 2.5

The proof of Theorem 2.5 is based on Theorem 3.1. We set

$$\mathcal{X} = \mathcal{X}(\Omega), \qquad \boldsymbol{X} = \boldsymbol{X}(\Omega), \qquad \boldsymbol{F}_0 = \mathbf{H}^3(\Omega), \qquad \boldsymbol{F}_1 = \mathbf{H}^1(\Omega).$$

Let $\boldsymbol{Y}$ be the closure of $\mathcal{X}$ in the space $\boldsymbol{F}_0$.

Define the operators $\boldsymbol{T}_i \colon \boldsymbol{X} \to \boldsymbol{Z}_i^*$, $\boldsymbol{Q}_i \colon \boldsymbol{X} \times \boldsymbol{Y} \to \boldsymbol{Z}_i$ by the following formulas:

$$\begin{aligned}
\langle \boldsymbol{T}_1(\boldsymbol{v}), \boldsymbol{\psi} \rangle &= -\left(\boldsymbol{v}, \boldsymbol{\psi}\right), \\
\boldsymbol{Q}_1[\boldsymbol{v}, \boldsymbol{\varphi}] &= \sum_{j=1}^{n} v_j \frac{\partial \boldsymbol{\varphi}}{\partial x_j}, \\
\langle \boldsymbol{T}_2(\boldsymbol{v}), \boldsymbol{\psi} \rangle &= \frac{\mu}{2}\left(\boldsymbol{A}_1(\boldsymbol{v}), \boldsymbol{\psi}\right), \\
\boldsymbol{Q}_2[\boldsymbol{v}, \boldsymbol{\varphi}] &= \boldsymbol{A}_1(\boldsymbol{\varphi}), \\
\langle \boldsymbol{T}_3(\boldsymbol{v}), \boldsymbol{\psi} \rangle &= \frac{\alpha}{2}\left(\boldsymbol{A}_1(\boldsymbol{v}), \boldsymbol{\psi}\right), \\
\boldsymbol{Q}_3[\boldsymbol{v}, \boldsymbol{\varphi}] &= \sum_{j=1}^{n} v_j \frac{\partial \boldsymbol{A}_1(\boldsymbol{\varphi})}{\partial x_j}, \\
\langle \boldsymbol{T}_4(\boldsymbol{v}), \boldsymbol{\psi} \rangle &= \frac{\alpha}{2}\big(\mathbf{A}_1(\boldsymbol{v})\mathbf{W}_\rho(\boldsymbol{v}) - \mathbf{W}_\rho(\boldsymbol{v})\mathbf{A}_1(\boldsymbol{v}), \mathbf{A}_1(\boldsymbol{\psi})\big), \\
\boldsymbol{Q}_4[\boldsymbol{v}, \boldsymbol{\varphi}] &= \boldsymbol{A}_1(\boldsymbol{\varphi}), \\
\langle \boldsymbol{T}_5(\boldsymbol{v}), \boldsymbol{\psi} \rangle &= \int\limits_{\Gamma} \lambda(\boldsymbol{x}, |\boldsymbol{v}_\tau|)(\boldsymbol{v}_\tau \cdot \boldsymbol{\psi}_\tau)\, d\Gamma, \\
\boldsymbol{Q}_5[\boldsymbol{v}, \boldsymbol{\varphi}] &= \boldsymbol{\varphi}.
\end{aligned}$$

It is clear that the weak statement of problem (2.1)–(2.4) is equivalent to the following equation

$$\sum_{i=1}^{5} \langle \boldsymbol{T}_i(\boldsymbol{v}), \boldsymbol{Q}_i[\boldsymbol{v}, \boldsymbol{\varphi}] \rangle = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle, \qquad \boldsymbol{\varphi} \in \mathcal{X}.$$

Trivially, the operators $\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_5$ satisfy condition (i).

Using the equalities

$$\sum_{j=1}^{n} \left(v_j \boldsymbol{v}, \frac{\partial \boldsymbol{v}}{\partial x_j} \right) = 0, \qquad \sum_{j=1}^{n} \left(\mathbf{A}_1(\boldsymbol{v}), v_j \frac{\partial \mathbf{A}_1(\boldsymbol{v})}{\partial x_j} \right) = 0,$$

$$\big(\mathbf{A}_1(\boldsymbol{v})\mathbf{W}_\rho(\boldsymbol{v}) - \mathbf{W}_\rho(\boldsymbol{v})\mathbf{A}_1(\boldsymbol{v}), \mathbf{A}_1(\boldsymbol{v})\big) = 0,$$

we get

$$\langle \boldsymbol{T}_1(\boldsymbol{v}), \boldsymbol{Q}_1[\boldsymbol{v},\boldsymbol{v}]\rangle = 0, \qquad \langle \boldsymbol{T}_3(\boldsymbol{v}), \boldsymbol{Q}_3[\boldsymbol{v},\boldsymbol{v}]\rangle = 0, \qquad \langle \boldsymbol{T}_4(\boldsymbol{v}), \boldsymbol{Q}_4[\boldsymbol{v},\boldsymbol{v}]\rangle = 0 \tag{4.1}$$

for any $\boldsymbol{v} \in \mathcal{X}$.

Let the function a be given by $r \to a(r) = r^2$. Obviously, the function a satisfies condition (iii). Taking into account (4.1), we obtain

$$\sum_{i=1}^{5} \langle \boldsymbol{T}_i(\boldsymbol{v}), \boldsymbol{Q}_i[\boldsymbol{v},\boldsymbol{v}]\rangle \geq a(\|\boldsymbol{v}\|_X), \qquad \boldsymbol{v} \in \mathcal{X}.$$

Now we must only prove that condition (iv) holds. Let $\boldsymbol{v}^k \to \boldsymbol{v}^0$ weakly in $\boldsymbol{X}$. Since the embedding $\boldsymbol{H}^1(\Omega) \subset \mathbf{L}_4(\Omega)$ is compact (see e.g. [1]), we see that $\boldsymbol{v}^k \to \boldsymbol{v}^0$ strongly in $\mathbf{L}_4(\Omega)$. Therefore, for $i = 1, 2, 3$, condition (iv) holds.

Note that

$$\begin{aligned}
\|(\mathbf{W}_\rho)_{jk}(\boldsymbol{v})\|_{L_\infty(\Omega)} &= \frac{1}{2} \sup_{x\in\Omega} \left| \int\limits_{\mathbb{R}^n} \rho(\boldsymbol{x}-\boldsymbol{y}) \left(\frac{\partial v_j(\boldsymbol{y})}{\partial y_k} - \frac{\partial v_k(\boldsymbol{y})}{\partial y_j} \right) d\boldsymbol{y} \right| \\
&= \frac{1}{2} \sup_{x\in\Omega} \left| \int\limits_{\mathbb{R}^n} -\frac{\partial \rho(\boldsymbol{x}-\boldsymbol{y})}{\partial y_k} v_j(\boldsymbol{y}) + \frac{\partial \rho(\boldsymbol{x}-\boldsymbol{y})}{\partial y_j} v_k(\boldsymbol{y})\, d\boldsymbol{y} \right| \\
&\leq \|\nabla\rho\| \, \|\boldsymbol{v}\|_{\mathbf{L}_2(\Omega)}.
\end{aligned}$$

This yields that condition (iv) is true for $i = 4$.

Finally note that the operator $\gamma_0 \colon \mathbf{H}^1(\Omega) \to \mathbf{L}_2(\Gamma)$ is compact. Hence, $\boldsymbol{v}^k|_\Gamma \to \boldsymbol{v}^0|_\Gamma$ strongly in $\mathbf{L}_2(\Gamma)$. By Krasnoselskii's theorem [15,24] on continuity of the Nemytskii operator, we have

$$\lambda(\cdot, |\boldsymbol{v}_\tau^k|)\boldsymbol{v}_\tau^k \to \lambda(\cdot, |\boldsymbol{v}_\tau^0|)\boldsymbol{v}_\tau^0 \quad \text{strongly in } \mathbf{L}_2(\Gamma).$$

It follows that condition (iv) is true for $i = 5$. This completes the proof.

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